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Cluster Approximation for the Farey Fraction Spin Chain

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We consider the Farey fraction spin chain in an external field h. Utilising ideas from dynamical systems, the free energy of the model is derived by means of an effective cluster energy approximation. This approximation is valid for divergent cluster sizes, and hence appropriate for the discussion of the magnetizing transition. We calculate the phase boundaries and the scaling of the free energy. At $h = 0$ we reproduce the rigorously known asymptotic temperature dependence of the free energy. For $h \neq 0$, our results are largely consistent with those found previously using mean field theory and renormalization group arguments.

KEY WORDS: Phase transition, Farey fractions, spin chain, cluster approximation

1. INTRODUCTION

The Farey fraction spin chain, which we study here, is one example of a set of closely related one-dimensional models (see ref. 1, 2 for details) which are of interest in both statistical mechanics and number theory. From the statistical mechanics point of view, there are a number of results, mainly for $h = 0$. All the models have the same free energy $f(\beta, h = 0)$, and exhibit a phase transition, at finite temperature, that is rather unusual. It is known, rigorously, to lie on the border between first- and second-order with the asymptotic form $f(\beta, 0) \sim t/\log t$ (where $t = 1 - \beta/\beta_c$). In the low-temperature state $f(\beta, 0) = 0$ and the

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magnetization is saturated, so there are no thermal effects at all. For $\beta < \beta_c$, $f(\beta, 0) < 0$ and the magnetization vanishes (see ref. 3 for details).

As the model possesses a non-trivial phase transition even in zero-field, it is physically interesting to investigate the effect of the coupling to an external field. In ref. 1 it is proven that the saturated state persists for $h \neq 0$ and $\beta > \beta_c$; when $h > 0$, the magnetization $m = 1$ and when $h < 0$, $m = -1$. The interesting question is how these states relate to the high-temperature state. In ref. 1 this question is addressed via a renormalization group calculation which finds, among other results, a phase diagram that is the same as illustrated in Fig. 3 below. However, these models have long-range interactions, so the applicability of the renormalization group might be questioned. It was in fact the desire to verify the results at non-zero h in ref. 1 that motivated this work (the results of ref. 1 for $h = 0$ agree with the rigorous behavior). Interestingly, the two approaches lead to subtly different results, as discussed below.

The Farey fraction spin chain has also led to some new results in number theory.[4–6] Additionally, as explained below, there is a connection to dynamical systems—in fact, the asymptotic form of $f(\beta, 0)$ mentioned comes from a result for dynamical systems.[7] Furthermore, chaotic behavior is even exhibited by certain statistical quantities. In particular, ref. 5 proves that the "density of states" for the infinite chain does not exist—it is a distribution.

In what follows, we analyze the Farey fraction spin chain in a particular approximation, in which the energy of a configuration is described by single cluster energies. Within this approximation, the model becomes exactly solvable (and is closely related to the "necklace" or "bead" models of Fisher and Felderhof.[8] Furthermore, our results for $h \neq 0$ agree, in the main, with a previous analysis[1] that makes use of mean field theory and the renormalization group. There are, however, some intriguing differences.

The same analysis can also be applied to a whole class of models with the same free energy,[9,10] although some care may have to be taken when introducing an external field.

The Farey fraction spin chain[3,15] may be defined as a chain of N spins σ_i; $i = 1, 2, \ldots, N$ with two possible states $\sigma_i \in \{\uparrow, \downarrow\}$. Using the matrices

$$A_\uparrow = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad A_\downarrow = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{1}$$

we define the energy of a configuration of N spins $\{\sigma_i\}$ as

$$E_N(\{\sigma_i\}, h) = \log\Bigl(\mathrm{Tr} \prod_{i=1}^{N} A_{\sigma_i}\Bigr) - h \sum_{i=1}^{N} (\chi_\uparrow(\sigma_i) - \chi_\downarrow(\sigma_i)), \tag{2}$$

where $\chi_\uparrow(\sigma_i) = 1$ (0) for $\sigma_i = \uparrow$ ($\downarrow$) so that it counts the number of up spins; $\chi_\downarrow(\sigma_i)$ is defined similarly to count the number of down spins. The cyclic invariance of

the trace in (2) makes the system translationally invariant. The partition function is then given as a sum over 2^N spin configuration

$$Z_N(\beta, h) = \sum_{\{\sigma_i\}} e^{-\beta E_N(\{\sigma_i\},h)} . \tag{3}$$

Our focus is the limiting free energy

$$-\beta f(\beta, h) = \lim_{N\to\infty} \frac{1}{N} \log Z_N(\beta, h). \tag{4}$$

The paper is organized as follows. In Sec. 2 we utilize the thermodynamic formalism to give a dynamical systems interpretation of the Farey fraction spin chain and describe how this connection may be used to obtain an effective cluster approximation. This approximation is obtained by replacing the smooth dynamical system by a piecewise linear map which captures the essential features of the dynamics. This linearization is usually referred to as Gaspard–Wang linearization[11] and has a long history.[12–14]

The analysis of the Farey fraction spin chain within the cluster approximation is described in Sec. 3, leading to explicit equations for the free energy and the phase boundaries. Section 4 contains the calculation of the scaling properties near the critical point. A summary and comparison with the results from ref. 1 are contained in Sec. 5. Appendix contains the derivation of the asymptotics of the cluster partition function.

The remainder of this section deals with a reformulation of the model in terms of clusters of consecutive spins of equal state and closes with a brief discussion of our strategy for calculating $f(\beta, h)$ in the cluster approximation.

Iteration of the matrices $A_\uparrow$ and $A_\downarrow$ leads to

$$A_\uparrow^n = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} \quad \text{and} \quad A_\downarrow^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}. \tag{5}$$

One notices that while some matrix elements increase in size, the zero field energy for the associated configurations remains constant, as $\mathrm{Tr}A_\uparrow^n = \mathrm{Tr}A_\downarrow^n = 2$. These two states, in fact, are the ground states at zero field. (They are also responsible for the low temperature thermodynamics.[15])

The energy is increased considerably, however, once a *change of spin* occurs. It therefore is useful to think of a general configuration as a sequence of clusters of consecutive spins of equal state (irrespective of whether the state is $\uparrow$ or $\downarrow$, as in zero field the energy is invariant under spin flip). If there is no change of spin at all, one has (as mentioned) $Tr(\prod_{i=1}^N A_{\sigma_i}) = 2$. Once there is a change of spin, we can take advantage of the cylic invariance of the trace to make the configuration begin with $A_\uparrow$ (resp. $A_\downarrow$) and end with $A_\downarrow$ (resp. $A_\uparrow$). Thus the total number of spin changes $2K$ must be even (with $K \geq 1$), and we can describe such a spin configuration $\{\sigma_i\}_{i=1}^N$ by a sequence of $2K$ clusters of size $n_k \geq 1$ with

$\sum_{k=1}^{2K} n_k = N$. Therefore, using

$$A_\downarrow = SA_\uparrow S^{-1} \quad \text{with} \quad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = S^{-1} \tag{6}$$

we can write for any configuration with $K \geq 1$

$$\operatorname{Tr} \prod_{i=1}^{N} A_{\sigma_i} = \operatorname{Tr} \prod_{k=1}^{2K} M_{n_k} \quad \text{with} \quad M_{n_k} = A_\uparrow^{n_k} S = \begin{pmatrix} 0 & 1 \\ 1 & n_k \end{pmatrix}. \tag{7}$$

Let us now suppose that we could find a meaningful approximation of the form

$$\operatorname{Tr} \prod_{k=1}^{2K} M_{n_k} \approx \prod_{k=1}^{2K} e^{\epsilon_{n_k}} \tag{8}$$

for some choice of the ϵ_{n_k}. In this case, the energy simplifies considerably and (for all but the ground states) we can write

$$E_N(\{\sigma_i\}, h) \approx \sum_{k=1}^{2K} (\epsilon_{n_k} - (-1)^{k-1} h n_k). \tag{9}$$

Since the trace only factorizes for commuting matrices it is clear that introducing ϵ_{n_k} is an approximation. However, in the next section we use ideas from dynamical systems to construct such an approximation, which reproduces the rigorously known free-energy asymptotics near the phase transition in the field-free case.

Working in the grand canonical ensemble, we show in Sec. 3 that one obtains from this approximation an exact expression for the limiting free energy $f(\beta, h)$, given by

$$\Lambda\left(e^{\beta(f-h)}, \beta\right) \Lambda\left(e^{\beta(f+h)}, \beta\right) = 1, \tag{10}$$

with the cluster generating function

$$\Lambda(z, \beta) = \sum_{n=1}^{\infty} z^n e^{-\beta \epsilon_n}. \tag{11}$$

It follows from the approximation discussed in Sec. 3 that this generating function has a radius of convergence of 1. Hence for $\beta > \beta_c(h)$ the limiting free energy is $f(\beta, h) = -|h|$, which for $\beta > \beta_c(0)$ agrees with the rigorously known result in ref. 1. Thus the phase boundary $\beta_c(h)$ is given by

$$\Lambda(1, \beta_c) \Lambda(e^{-2\beta_c |h|}, \beta_c) = 1. \tag{12}$$

In particular, for zero field $h = 0$, the free energy $f(\beta, 0)$ and critical temperature $\beta_c(0)$ follow from

$$\Lambda(e^{\beta f}, \beta) = 1 \quad \text{and} \quad \Lambda(1, \beta_c) = 1, \tag{13}$$

respectively.

2. THERMODYNAMIC FORMALISM AND CLUSTER APPROXIMATION

In order to proceed further, we consider the thermodynamics of the Farey tree.[16] This can be recast in a transfer operator formulation associated with the iteration of an interval map (see refs. 1, 17). By modifying this interval map we arrive at the desired cluster approximation. Note that the Farey tree is known, rigorously, to have the same free energy $f(\beta, h = 0)$ as the Farey fraction spin chain (with no external field).[3]

The Farey tree is generated by the Farey map defined on the unit interval [0, 1], which is defined as

$$f(x) = \begin{cases} f_0(x) = x/(1-x)\,, & \text{if } 0 \le x \le 1/2, \\ f_1(x) = (1-x)/x\,, & \text{if } 1/2 < x \le 1, \end{cases} \tag{14}$$

(see Fig. 1). We denote the inverses by $F_0(x) = {f_0}^{-1}(x) = x/(1+x)$ and $F_1(x) = {f_1}^{-1}(x) = 1/(1+x)$. The associated transfer operator is formally given by

$$\begin{aligned} \mathcal{L}_\beta\ \phi(x) &= |F_0{}'(x)|^\beta \phi(F_0(x)) + |F_1{}'(x)|^\beta \phi(F_1(x)) \\ &= \frac{1}{(1+x)^{2\beta}} \left[\phi\left(\frac{x}{1+x}\right) + \phi\left(\frac{1}{1+x}\right) \right]. \end{aligned} \tag{15}$$

Therefore, the N-fold iterated operator $\mathcal{L}_\beta^N \phi(x)$ consists of 2^N terms of the form

$$|(F_{\tau_1} \circ F_{\tau_2} \circ \ldots, \circ F_{\tau_N})'(x)|^\beta \phi(F_{\tau_1} \circ F_{\tau_2} \circ \ldots \circ F_{\tau_N}(x)) \tag{16}$$

with $\tau_i \in \{0, 1\}$. As we are dealing with iterations of Möbius transformations of the form $\frac{ax+b}{cx+d}$ with determinant ± 1, we can alternatively consider multiplication of the associated matrices. (In a slight abuse of notation, we shall denote the Möbius transformation and the associated matrix by the same symbol.) We find for instance

$$\mathcal{L}_\beta^N 1(0) = \sum_{\{\tau_i\}} d_{\{\tau_i\}}^{-2\beta}, \tag{17}$$

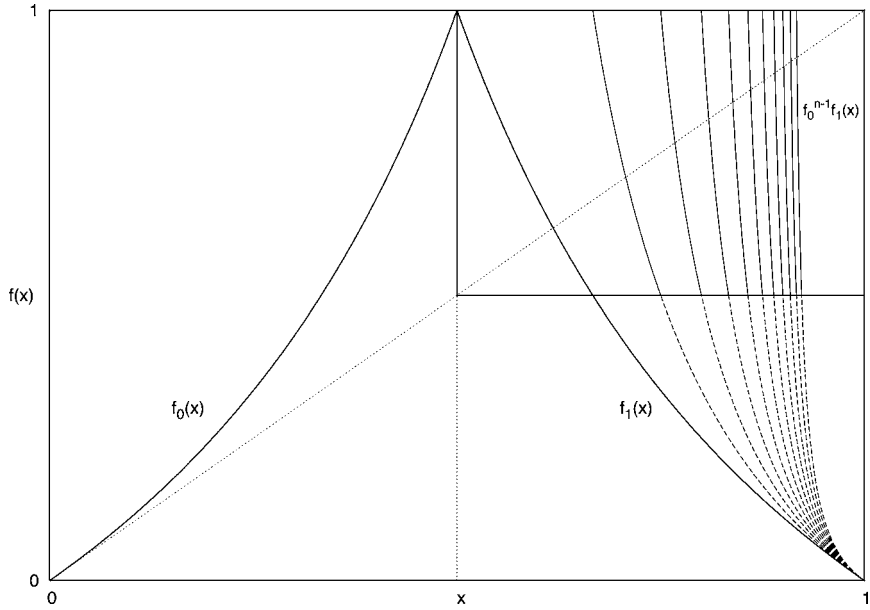

Fig. 1. Farey map and first-return map on the interval $[1/2, 1]$. The first-return map is given explicitly by the branches ${f_0}^{n-1} f_1$ for $n \in \mathbb{N}$. Their extension to all of $[1/2, 1]$ is shown by dashed lines.

where $d_{\{\tau_i\}}$ is just the bottom right entry of the matrix product

$$\prod_{i=1}^{N} F_{\tau_i} \quad \text{where} \quad F_0 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad F_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}. \tag{18}$$

Now

$$F_0^{n-1} F_1 = \begin{pmatrix} 0 & 1 \\ 1 & n \end{pmatrix} = M_n \tag{19}$$

which immediately suggests a cluster approximation analogous to that for the Farey model discussed in Sec. 1.

There are differences with the Farey fraction spin chain, however. One of them concerns details in the allowed clusterings. Both the Farey fraction spin chain partition function and the N-fold iterated transfer operator are expressible in sums containing 2^N terms. However, in the clustering representation of the Farey fraction spin chain one needs to take the cyclicity of the trace into account. This means that in the Farey fraction spin chain all excited states contain an even number of clusters. These issues (including multiplicity) are dealt with in Sec. 3.

More importantly, the energies are different. Note that the matrices F_0 and F_1 used to generate the $d_{\{\tau_i\}}$ in (17) differ from the matrices $A_\uparrow$ and $A_\downarrow$ employed in

the Farey fraction spin chain. Therefore $d_{\{\tau_i\}}$ in (17) cannot be compared directly with the trace (which is given in a similar way by $a_{\{\sigma_i\}} + d_{\{\sigma_i\}}$) in (2). Despite this, both expressions lead to the same free energy $f(\beta, h = 0)$. This was already known via the argument in ref. 3 that the largest eigenvalue of $\mathcal{L}_\beta$ is $e^{2\beta f(2\beta)}$, and the corresponding eigenvector is positive. More recently, the connection has been shown to be more direct. Ref. 2 proves that (see (17)) $\mathcal{L}_\beta^N 1(0) = 2Z_{N-1}^K(2\beta)$, where Z_N^K is the partition function of the Knauf spin chain, which is rigorously known[3] to have the same free energy $f(\beta, h = 0)$ as the Farey fraction spin chain (with no external field). Therefore it is reasonable to use the Farey tree cluster energies (see (24)) in a cluster approximation for the Farey fraction spin chain.

We now come to the main point of this section, that one can construct a piecewise linear version of the Farey map which captures its essential features while being significantly easier to analyse. This is done by linearizing the map between the inverse images $F_0^k(1/2)$ (see Fig. 2). This sequence tends to zero, so that the structure of the map near the fixed point at zero is preserved under linearization. Due to this fact the critical dynamical properties of the linearized map and the Farey map are still closely related[11]. In fact, the spectral radius of both associated transfer operators shows the same type of singular behavior at β_c.[19]

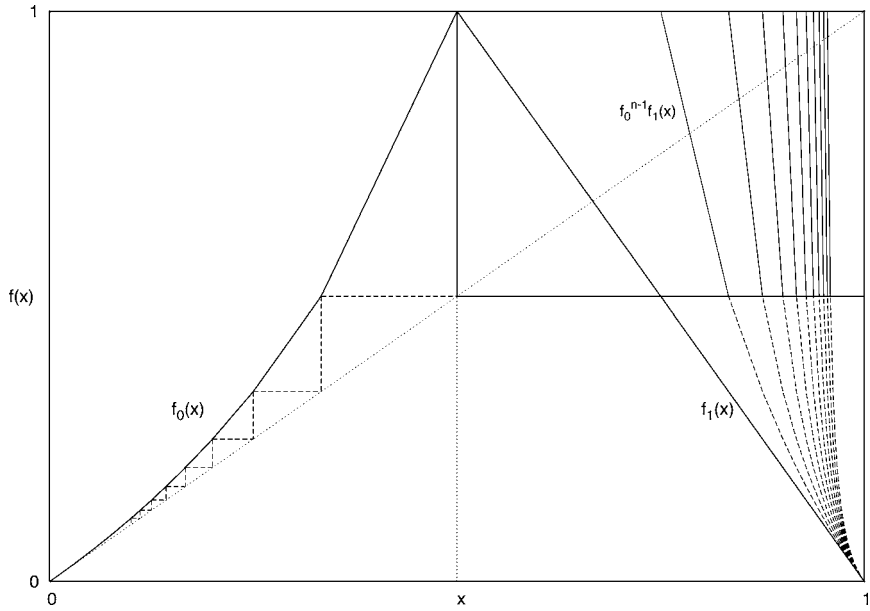

Fig. 2. Linearized Farey map and first-return map on the interval $[1/2, 1]$. The first-return map is given explicitly by the branches $f_0{}^{n-1} f_1$ for $n \in \mathbb{N}$, which are linear maps onto $[1/2, 1]$. Their extension to all of $[1/2, 1]$ is shown in dashed lines.

Next, we consider the first-return map on the interval $J = [1/2, 1]$, which is the map obtained by repeated iteration of f until $f^n(x)$ lies again in J. This map is the key to understanding intermittency in the Farey map.[7,19–21] The first-return map also forms the motivation for a comprehensive operator-theoretic treatment in ref. 22. Furthermore, it is responsible for making the particular piecewise linear map that we use easier to analyse than the original smooth one. If we define $n(x) = \min\{n \geq 1 | f^n(x) \in J\}$ then the first-return map can be written as $g(x) = f^{n(x)}(x)$. Due to our particular choice of J the first-return map g becomes particularly simple. Its branches are given by ${f_0}^{n-1} f_1$ for $n \in \mathbb{N}$ (see Fig. 1). One can define a (suitably modified) transfer operator $\mathcal{M}_{z,\beta}$ for the first-return map, given by

$$\mathcal{M}_{z,\beta}\phi(x) = \sum_{n=1}^{\infty} z^n \left|\left(F_1 {F_0}^{n-1}\right)'(x)\right|^\beta \phi\left(F_1 {F_0}^{n-1}(x)\right). \tag{20}$$

As the first-return map is only defined on J, this operator acts on functions with domain J. However, using the conjugacy $\mathcal{C}\phi(x) = |{f_1}'(x)|^\beta \phi(f_1(x))$ we obtain an equivalent conjugate operator $\mathcal{C}^{-1}\mathcal{M}_{z,\beta}\mathcal{C}$ acting on functions with domain $[0, 1]$ and given by

$$\mathcal{C}^{-1}\mathcal{M}_{z,\beta}\mathcal{C}\phi(x) = \sum_{n=1}^{\infty} z^n \left|\left({F_0}^{n-1} F_1\right)'(x)\right|^\beta \phi\left({F_0}^{n-1} F_1(x)\right), \tag{21}$$

which allows us to make the identification with $M_n = {F_0}^{n-1} F_1$.

The crucial observation is that the operator spectra of $\mathcal{L}_\beta$ and $\mathcal{M}_{z,\beta}$ are related,[7,19] in the sense that $\lambda = z^{-1}$ is an eigenvalue of $\mathcal{L}_\beta$ if and only if 1 is an eigenvalue of $\mathcal{M}_{z,\beta}$ (for a rigorous formulation see refs. 19, 22).

The important consequence of the particular linearization chosen is that the piecewise linearised map replaces the first-return map on J by a first-return map with branches $f_0^{n-1} f_1(x)$ which are linear and onto, with slopes $n(n+1)$. It follows that the eigenfunction associated with the leading eigenvalue $\Lambda(z, \beta)$ of $\mathcal{M}_{z,\beta}$ becomes constant, and that this eigenvalue is given explicitly by

$$\Lambda(z, \beta) = \sum_{n=1}^{\infty} \frac{z^n}{(n(n+1))^\beta}. \tag{22}$$

Note that $\Lambda(z, \beta)$ extends to an analytic function in the complex z-plane cut from 1 to infinity for all β.

Reformulated in terms of approximations to iterates of the transfer operator $\mathcal{L}_\beta$ (17), the linearization implies the approximation

$$\mathcal{L}_\beta^N 1(0) \approx \sum_{\sum_{k=1}^{2K} n_k = N} \frac{1}{(n_k(n_k+1))^\beta}, \tag{23}$$

and we recognize that the energy has become a sum of cluster energies

$$\epsilon_n = \log n(n+1). \tag{24}$$

In other words, the leading eigenvalue (22) of the modified transfer operator of the linearized Farey map is identical with the cluster generating function obtained from (11) and (24).

We have mapped the thermodynamics of the Farey tree to a dynamical system given by an interval map, and introduced a linearization of this interval map. In this way we obtain a cluster energy approximation to the original energy expression. This replaces the original model, which is very difficult to work with, with an approximation that is quite tractable. In the next section, we apply this cluster approximation to the Farey fraction spin chain in a field and determine its consequences.

3. THERMODYNAMICS IN THE CLUSTER APPROXIMATION

As discussed in the introduction, the Farey partition function can be written in terms of the cluster representation $\{n_k\}_{k=1}^{2K}$. There are two ground states with all N spins either up or down. The excited states consist of configurations with $2K$ alternating clusters of spins with length $n_k \geq 1$. This leads to a degeneracy n_1, since (as discussed) we insist that the first and last clusters must have opposite spin directions.

Now the first cluster of size n_1 has either all spins up or all spins down. It is therefore convenient to split the partition function into two terms, according to the state of the first spin. If we denote the partition function of configurations with $\sigma_1 = \uparrow$ by $Z_N^\uparrow(\beta, h)$ and the partition function of configurations with $\sigma_1 = \downarrow$ by $Z_N^\downarrow(\beta, h)$, then $Z_N^\downarrow(\beta, h) = Z_N^\uparrow(\beta, -h)$ and we obtain

$$Z_N^\uparrow(\beta, h) = (2e^{hN})^{-\beta} + \sum_{\substack{\sum_{k=1}^{2K} n_k = N \\ K \in \{1,\ldots,\lfloor N/2 \rfloor\}}} n_1 \Big(\prod_{k=1}^{2K} e^{(-1)^{k-1} h n_k} \operatorname{Tr} \prod_{k=1}^{2K} M_{n_k} \Big)^{-\beta}. \tag{25}$$

The partition function can then be obtained as

$$Z_N(\beta, h) = Z_N^\uparrow(\beta, h) + Z_N^\downarrow(\beta, h) = Z_N^\uparrow(\beta, h) + Z_N^\uparrow(\beta, -h). \tag{26}$$

Using the approximation discussed in Sec. 2 leads to

$$Z_N^{\uparrow}(\beta, h) \approx (2e^{hN})^{-\beta} + \sum_{\sum_{k=1}^{2K} n_k = N} n_1 \prod_{k=1}^{2K} e^{-\beta(\epsilon_{n_k} + (-1)^{k-1} h n_k)}. \tag{27}$$

where now

$$\epsilon_n = \frac{1}{2} \log n(n+1), \tag{28}$$

with the factor $1/2$ arising from (17). (Note that the energy (28) of a cluster is, for large n, the same as the energy of the lowest excited state of a chain of length n.) From this, one can determine the limiting free energy $f(\beta, h)$. Note also that this approximation does not change β_c. This is because the change in spectral radius of the transfer operator occurs at the same temperature after linearization.[11,19]

Passing to the grand canonical ensemble, we write

$$G^{\uparrow}(z, \beta, h) = \sum_{N=1}^{\infty} z^N Z_N^{\uparrow}(\beta, h). \tag{29}$$

$G^{\downarrow}(z, \beta, h)$ and $G(z, \beta, h)$ are then defined similarly. One finds

$$\begin{aligned} G^{\uparrow}(z, \beta, h) \;&\approx\; 2^{-\beta} \frac{z e^{-\beta h}}{1 - z e^{-\beta h}} + \sum_{K=1}^{\infty} \sum_{n_1, \ldots, n_{2K} = 1}^{\infty} n_1 \prod_{k=1}^{2K} (z e^{(-1)^k \beta h})^{n_k} e^{-\beta \epsilon_{n_k}} \\ &= 2^{-\beta} \frac{z e^{-\beta h}}{1 - z e^{-\beta h}} + \sum_{K=1}^{\infty} \sum_{n_1=1}^{\infty} n_1 (z e^{-\beta h})^{n_1} e^{-\beta \epsilon_{n_1}} \ldots \\ &\ldots \sum_{n_{2K}=1}^{\infty} (z e^{\beta h})^{n_{2K}} e^{-\beta \epsilon_{n_{2K}}}. \end{aligned} \tag{30}$$

As in the introduction, we define the cluster generating function

$$\Lambda(z, \beta) = \sum_{n=1}^{\infty} z^n e^{-\beta \epsilon_n}. \tag{31}$$

After some transformations we arrive at (note the close resemblance to the results in Sec. 6 of ref. 8—one difference being that the parameter ψ is a function of β here)

$$G^{\uparrow}(z, \beta, h) \approx 2^{-\beta} \frac{z e^{-\beta h}}{1 - z e^{-\beta h}} + \frac{z e^{-\beta h} \partial_1 \Lambda(z e^{-\beta h}, \beta) \Lambda(z e^{\beta h}, \beta)}{1 - \Lambda(z e^{-\beta h}, \beta) \Lambda(z e^{\beta h}, \beta)}. \tag{32}$$

This is perhaps most easily seen by expanding Eq. (32) backwards. The denominator in the second term corresponds to the sum over n_1 and n_2. The partial derivative

leads to the multiplying factor n_1. Expanding the denominator into a geometric series produces the product over the sums with summation index n_i with $i \geq 3$. We now get

$$G(z, \beta, h) = G^{\uparrow}(z, \beta, h) + G^{\downarrow}(z, \beta, h), \tag{33}$$

where $G^{\downarrow}(z, \beta, h) = G^{\uparrow}(z, \beta, -h)$. The limiting free energy is then given as

$$\beta f(\beta, h) = \log z_c(\beta, h), \tag{34}$$

where $z_c(\beta, h)$ is the smallest singularity of $G(z, \beta, h)$ for z on the positive real axis. This singularity is reached at the smallest positive solution z_c of one of the three equations

$$ze^{-\beta h} = 1\ , \quad ze^{\beta h} = 1\ , \quad \Lambda(ze^{-\beta h}, \beta)\Lambda(ze^{\beta h}, \beta) = 1. \tag{35}$$

The first two equations correspond to the two fully magnetized phases, where $z_c = e^{-\beta|h|}$ or, in terms of the free energy, $f = -|h|$. The third equation corresponds to the high-temperature phase. In terms of the free energy,

$$\Lambda(e^{\beta(f-h)}, \beta)\Lambda(e^{\beta(f+h)}, \beta) = 1. \tag{36}$$

Since the cluster generating function $\Lambda(z, \beta)$ has radius of convergence $z = 1$, the phase boundary is given by $f = -|h| = -h_c$, (which means that the phase boundary in fact follows from the last equation of (35)), and $h_c(\beta)$ is determined by

$$\Lambda(1, \beta)\Lambda(e^{-2\beta h_c}, \beta) = 1. \tag{37}$$

The three phases meet at a critical point given by $h = 0$ and $\Lambda(1, \beta_c) = 1$, i.e. $\beta_c = 2$.

4. SCALING OF THE FREE ENERGY

In this section we calculate the asymptotic behavior of the solutions $f(\beta, h)$ and $h_c(\beta)$ of (36) and (37), respectively, near the critical point, which is given by $h = 0$ and $\beta = \beta_c$. For this we need the asymptotic behavior of the cluster partition function for $\beta \to \beta_c$ and $z \to 1$. Introducing the reduced temperature $t = 1 - \beta/\beta_c$, we find

$$\Lambda(z, \beta) \sim 1 + Ct + (1 - z)\log(1 - z) \tag{38}$$

(a derivation is given in Appendix). Inserting $z = e^{\beta(f \pm h)}$ into (38) gives $1 - z \sim -\beta_c(f \pm h)$, and we obtain to leading order

$$\Lambda(e^{\beta(f \pm h)}, \beta) \sim 1 + Ct - \beta_c(f \pm h)\log[-\beta_c(f \pm h)]. \tag{39}$$

Thus, (36) implies that to leading order

$$2Ct \sim \beta_c(f + h)\log[-\beta_c(f + h)] + \beta_c(f - h)\log[-\beta_c(-f + h)]. \tag{40}$$

In the field free case ($h = 0$) this simplifies to

$$Ct \sim \beta_c f \log(-\beta_c f), \tag{41}$$

which can be inverted to give

$$f \sim \frac{C}{\beta_c} \frac{t}{\log t}. \tag{42}$$

For the phase boundary ($-f = |h| = h_c$) we find, using (37)

$$2Ct \sim -2\beta_c h_c \log(2\beta_c h_c), \tag{43}$$

which can be inverted to give

$$h_c \sim -\frac{C}{\beta_c} \frac{t}{\log t}. \tag{44}$$

(see Fig. 3). Note that this result implies that on the phase boundary to leading order $f(\beta, h_c(\beta)) \sim f(\beta, h = 0)$ as $\beta \to \beta_c$, and differences only appear in higher order terms.

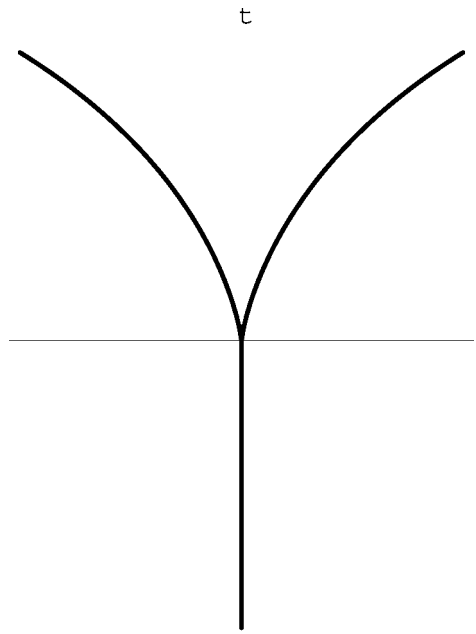

Fig. 3. Phase boundaries for the Farey fraction spin chain near the critical point, showing the disordered high-temperature phase (top) and the two fully magnetized phases (bottom left and right).

Finally, we compute the leading correction to the zero-field free energy in h. Assuming $|h| \ll -f$ (this assumption of course breaks down near the phase boundary) in (38) gives

$$2Ct \sim 2\beta_c f \log(-\beta_c f) + \beta_c h^2/f. \tag{45}$$

Inverting this finally leads to

$$f \sim \frac{C}{\beta_c}\frac{t}{\log t} - \frac{\beta_c}{2C}\frac{h^2}{t} \tag{46}$$

for $|h| \ll |t/\log t|$.

Equations (42), (44), and (46) are our main results. We discuss them in the next section.

5. SUMMARY AND DISCUSSION

In this paper we have calculated the free energy $f(\beta, h)$ of the Farey fraction spin chain in an external field h by making use of a cluster approximation for the energy of the excited state spin configurations.

We conclude with a summary of our results and comparison to previous work.

- In the case of zero field, we find

  $$f(\beta, h = 0) \sim \frac{C}{\beta_c}\frac{t}{\log t} \quad \text{where } t = 1 - \beta/\beta_c.$$

  The temperature dependence of this result agrees with the known rigorous result,[3] the renormalization group calculations,[1] and a rigorous analysis of the (non-linearized) Farey transfer operator.[17]
- For the phase boundary, where $h_c = |h| = -f$, we find

  $$h_c(\beta) \sim -\frac{C}{\beta_c}\frac{t}{\log t} \quad \text{where } t = 1 - \beta/\beta_c.$$

  The temperature dependence again agrees with the renormalization group calculations.[1] Additionally, we find that the constants in $f(\beta, 0)$ and $h_c(\beta)$ are equal, i.e. we have to leading order $h_c(\beta) \sim -f(\beta, h = 0)$, (where $h_c(\beta) = -f(\beta, h_c(\beta))$). This is beyond what renormalization group calculations can predict, since there are several undetermined constants in that case (see ref. 1 for details). What is more interesting is that this equality is not consistent with the renormalization group results, as explained below.
- The change of the free energy for small fields is given by

  $$f(\beta, h) \sim \frac{C}{\beta_c}\frac{t}{\log t} - \frac{\beta_c}{2C}\frac{h^2}{t} \quad \text{where } t = 1 - \beta/\beta_c \quad \text{and} \quad |h| \ll |t/\log t|.$$

This is not in accordance with renormalization group calculations. In that case one has[1]

$$f(\beta, h) \sim a\frac{t}{\log t} - b\frac{h^2 \log t}{t},$$

i.e. a correction term of the order of $h^2 \log t/t$, with a and b undetermined constants. Setting this expression equal to $-h_c$ in order to determine the phase boundary, one finds that if the constants in $f(\beta, 0)$ and h_c are equal, i.e. if the cluster results just mentioned hold, one must have $b = 0$. This is, in a sense, consistent, since it might imply that the leading correction to the free energy for finite h is of higher order than $h^2 \log t/t$, and therefore could indeed be h^2/t. However, it does not seem to be possible to alter the renormalization calculation to obtain this while keeping the correct form for $f(\beta, 0)$, the free energy at $h = 0$. Setting $b = 0$ implies that the parameter $x = 0$ in ref. 1. If one then includes a higher order term in the flow equation for u (Eq. (10) in ref. 1), the result for $f(\beta, 0)$ is no longer correct. Another possibility is that u is a "dangerous irrelevant variable" (or, more precisely, a "dangerous marginal variable"), and the finite-field correction term to the free energy takes the form $h(\ell_0)^2 u(\ell_0)/t(\ell_0)$ far from the critical point, which results in a leading contribution to f in agreement with the cluster result found herein. Since very little is understood about u, however, such an assumption is completely *ad hoc*. So the question of the correct leading term for the free energy at finite h remains open. Now, one might question the applicability of the renormalization group to this model, due to the presence of long-range forces. The cluster results, being more closely tailored to the Farey fraction spin chain, are more likely correct, but they are not rigorous either. Therefore, it would be interesting to know what the correct behavior is. A rigorous asymptotic analysis of the transfer operator for finite h appears possible,[23] and should answer this question.

- As mentioned in the Introduction, there is a set of closely related models, including the Farey fraction spin chain, which all have the same free energy $f(\beta, h = 0)$ at zero external field. In addition, the magnetization is the same (see ref. 3 for details). One therefore expects, according to scaling theory, that they have the same free energy for $h \neq 0$ as well. The analysis in ref. 1 makes this assumption. However, it has not been proven. Given the presence of long-range forces in these systems, one might doubt its validity. The results obtained here do support it, in that the renormalization group and cluster approximation approaches agree for the most part. However, as mentioned, the agreement is not perfect, and is in any case limited to one particular model (the Farey fraction spin chain). Therefore further work on these models seems called for.

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APPENDIX: ASYMPTOTIC ANALYSIS OF THE CLUSTER PARTITION FUNCTION

Here, we present the derivation of the asymptotic behavior (38) of the cluster partition function $\Lambda(z, \beta)$ near $(z, \beta) = (1, \beta_c)$, where the critical temperature β_c is determined by $\Lambda(1, \beta_c) = 1$.

We recall that the cluster partition function is given by

$$\Lambda(z, \beta) = \sum_{n=1}^{\infty} z^n e^{-\beta \epsilon_n}. \tag{A.1}$$

where in our case

$$\epsilon_n = \frac{1}{2} \log[n(n+1)], \tag{A.2}$$

and $\Lambda(1, 2) = 1$ implies $\beta_c = 2$. For $\beta = \beta_c$ one obtains

$$\Lambda(z, \beta_c) = 1 + \frac{1-z}{z} \log(1-z). \tag{A.3}$$

For $\beta > 1$ and $|z| \leq 1$, $\Lambda(z, \beta)$ is an analytic function in β with coefficients depending on z. In particular, expanding around $\beta = \beta_c$, we get

$$\begin{aligned} \Lambda(z, \beta) &= \Lambda(z, \beta_c) + \sum_{n=1}^{\infty} \frac{z^n}{n(n+1)} \left([n(n+1)]^{(\beta_c - \beta)/\beta_c} - 1 \right) \qquad \text{(A.4)} \\ &= \Lambda(z, \beta_c) + \frac{\beta_c - \beta}{\beta_c} \sum_{n=1}^{\infty} \frac{z^n}{n(n+1)} \log[n(n+1)] + O((\beta_c - \beta)^2) \end{aligned}$$

uniformly in $|z| \leq 1$. In particular, letting z approach one from below, the coefficient of the linear term in $(\beta_c - \beta)$ changes continuously with z and we arrive at

$$\Lambda(z, \beta) = \Lambda(z, \beta_c) + \frac{\beta_c - \beta}{\beta_c} (C + o((1-z)^0) + O((\beta_c - \beta)^2) \tag{A.5}$$

with $C = \sum_{n=1}^{\infty} \frac{\log[n(n+1)]}{n(n+1)} = 2.046277452855878591\ldots$ (the sum can easily be evaluated numerically using the Euler-MacLaurin formula). Introducing the

reduced temperature $t = 1 - \beta/\beta_c$, this implies

$$\Lambda(z, \beta) \sim 1 + Ct + (1 - z)\log(1 - z), \tag{A.6}$$

which is our desired Eq. (38).

We conclude this appendix with a few generalizing remarks. For our purposes it was sufficient to work directly with (22), but we would like to point out that it is possible to perform a more thorough analysis. Observing that

$$\frac{1}{[n(n+1)]^{\beta/2}} = \int_0^\infty K(\beta, s)e^{-ns}ds,$$

where

$$K(\beta, s) = \frac{\sqrt{\pi}}{\Gamma(\beta/2)} I_{(\beta-1)/2}(s/2)e^{-s/2}s^{(\beta-1)/2},$$

leads to

$$\Lambda(z, \beta) = \sum_{n=1}^\infty \frac{z^n}{[n(n+1)]^{\beta/2}} = \int_0^\infty K(\beta, s)\frac{z}{e^s - z}ds,$$

an integral representation which is a different starting point for an asymptotic analysis. In particular, one recognizes directly that the large-n asymptotics of the cluster energies ϵ_n is related to the small-s expansion of the integral kernel $K(\beta, s)$, which in turn determines the singular behavior of $\Lambda(z, \beta)$. The argument can therefore also be extended to more general ϵ_n.

REFERENCES

1. J. Fiala and P. Kleban, Thermodynamics of the Farey fraction spin chain, *J. Stat. Phys.* **110**:73–86 (2004) [arXiv: math-ph/0310016].
2. J. Fiala and P. Kleban, Generalized number theoretic spin chain–connections to dynamical systems and expectation values, *J. Stat. Phys.* to appear [arXiv: math-ph/0503030].
3. J. Fiala, P. Kleban and A. Özlük, The phase transition in statistical models defined on Farey fractions, *J. Stat. Phys.* **110**:73–86 (2003) [arXiv: math-ph/0203048].
4. J. Kallies, A. Özlük, M. Peter and C. Snyder, On asymptotic properties of a number theoretic function arising from a problem in statistical mechanics, *Commun. Math. Phys.* **222**:9–43 (2001).
5. M. Peter, The limit distribution of a number-theoretic function arising from a problem in statistical mechanics, *J. Number Theory* **90**:265–280 (2001).
6. F. Boca, *Products of matrices and and the distribution of reduced quadratic irrationals*, preprint [arXiv: math.NT/0503186].
7. T. Prellberg, *Maps of intervals with indifferent fixed points: thermodynamic formalism and phase transition*, Ph.D. thesis, Virginia Tech (1991).
8. M. E. Fisher Walks, walls wetting and melting, *J. Stat. Phys.* **34**:667–729 (1984).
9. A. Knauf, On a ferromagnetic spin chain, *Commun. Math. Phys.* **153**:77–115 (1993).
10. A. Knauf, The numbertheoretical spin chain and the Riemann zeros, *Commun. Math. Phys.* **196**:703–731 (1998).

11. P. Gaspard and X.-J. Wang, Sporadicity: Between Periodic and Chaotic Dynamical Behaviors, *Proc. Nat. Acad. Sci. USA* **85**:4591–4595 (1988).
12. A. Lambert, S. Siboni and S. Vaienti, Statistical properties of a non-uniformly hyperbolic map of the interval, *J. Stat. Phys.* **72**:1305–1330 (1993).
13. M. Mori, On the intermittency of a piecewise linear map, *Tokyo J. Math.* **16**:411–442 (1993).
14. S. Isola, Renewal sequences and intermittency, *J. Stat. Phys.* **97**:263–280 (1999).
15. P. Kleban and A. Özlük, A Farey fraction spin chain, *Commun. Math. Phys.* **203**:635–647 (1999).
16. M. J. Feigenbaum, I. Procaccia and T. Tél, Scaling properties of multifractals as an eigenvalue problem, *Phys. Rev. A* **39**:5359–5372 (1989).
17. T. Prellberg, Towards a complete determination of the spectrum of a transfer operator associated with intermittency, *J. Phys. A: Math. Gen.* **36**:2455–2461 (2003).
18. P. Contucci and A. Knauf, The phase transition of the number-theoretic spin chain, *Forum Mathematicum* **9**:547–567 (1997).
19. T. Prellberg and J. Slawny, Maps of intervals with indifferent fixed points: thermodynamic formalism and phase transition, *J. Stat. Phys.* **66**:503–514 (1992).
20. N. Haydn and S. Isola, Parabolic rational maps, *J. London Math. Soc.* **63**(2):673–689 (2001).
21. S. Isola, On the spectrum of Farey and Gauss maps, *Nonlinearity* **15**:1521–1539 (2002).
22. H. H. Rugh, Intermittency and regularized Fredholm determinants, *Invent. Math.* **135**:1–24 (1999).
23. T. Prellberg, P. Kleban and J. Fiala, in progress.